Derivation of the Kinetic Energy Equations

used by the

Revolving Door Energy Calculator (RDEC) Program

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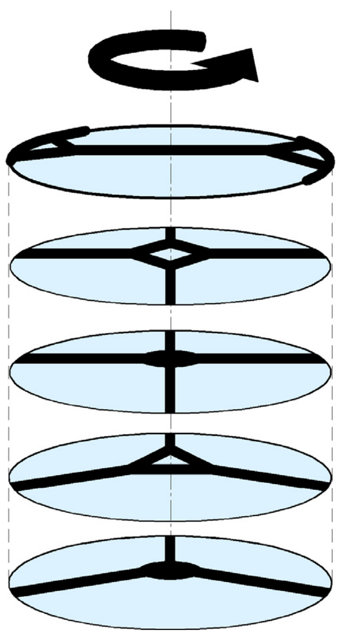

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CONTENTS

Introduction

The Revolving Door Energy Calculator (RDEC) program from Davis Associates, Inc., is used to calculate the rotational kinetic energy of revolving doors of the following five types:

- 2-wing doors with showcases
- 3-wing doors without a core
- 3-wing doors with a core
- 4-wing doors without a core
- 4-wing doors with a core

This document describes the mathematical derivation of the kinetic energy equations used by the RDEC program for each of the above five types of doors.

In addition to the kinetic energy equations that apply to each basic door type, RDEC also incorporates equations relating to significant features that also contribute to the kinetic energy that are currently not considered by the ANSI/BHMA A156.27 national standard for revolving doors. These include:

- A ceiling that co-rotates with the door
- Showcase floors
- Co-rotating objects with negligible self moment-of-inertia
- Co-rotating objects with non-negligible self moment-of-inertia

The derivations of the relevant kinetic energy equations are based on the laws of physics that apply to rigid body rotation. That is, to objects in which the locations and orientations of components maintain a constant relationship to each other as the object rotates. These laws have been known essentially since the time of Isaac Newton and, indeed, are contained within the corpus of physical knowledge known as Newtonian mechanics.

One of the primary tools used in the derivation of the kinetic energy equations herein is the so-called parallel axis, or Huygens–Steiner, theorem. This theorem states that the total rotational kinetic energy of an object participating in rigid body rotation is equal to the sum of two parts. The first is the kinetic energy corresponding to the entire mass of the object concentrated at the location of its own center-of-mass revolving around the axis of rotation. The second is the kinetic energy corresponding to the rotation of the object about its own center-of-mass. For use in the parallel axis theorem, this requires specification of the moment-of-inertia of the object about it own center-of-mass, otherwise referred to as the self moment-of-inertia of the object. The rate of rotation of the object about its own center-of-mass is identical to the overall rotation rate of the body within which the object is embedded due to the fact of rigid body rotation.

Given relationships for the self moments-of-inertia of the individual components of the revolving door, the derivation of the relevant kinetic energy equations is accessible to a high school AP physics student possessing only a pre-calculus and plane geometry mathematical background. In the case of the RDEC program, relationships for the self moments-of-inertia of most of the door components are widely available in handbooks of various kinds, and even in pre-calculus high school physics texts.[1] In one instance, the requisite self moments-of-inertia – those relating to the showcase floors associated with 2-wing revolving doors – are not readily available from other sources and, consequently, are derived in Appendices attached to this document. These derivations require integral calculus and consequently might, or might not, be accessible to a high school AP physics student.

Assumptions

The kinetic energy equations derived here and used by the RDEC program are based on the following assumptions and physical facts:

1. The moving components of the revolving door undergo rigid rotation. That is, they are rigidly ganged together so that they maintain fixed positions and orientations with respect to each other as the door rotates.

2. The axis of rotation of the door is vertical and is located at the geometric center of the assembly of moving door components.

3. The thickness of vertically oriented panels is a small fraction of the width of the panel.

4. The distribution of mass within flat vertically oriented panels is uniform in the horizontal direction across the width of such panels and is immaterial in the vertical direction.

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[1] See for example, *Physics: Principles with Applications*, Fifth Edition, Douglas C. Giancoli, Prentice Hall, NJ (1980). In particular, see Figure 8-20, p. 223.

5. The distribution of mass within curved panels all points of which are at a common, fixed horizontal distance from the axis of rotation is immaterial in both the horizontal and vertical directions.

6. The distribution of mass within horizontally oriented surfaces, such as a co-rotating ceiling and showcase floors, is uniform in the horizontal direction and immaterial in the vertical direction.

7. The thickness of horizontally oriented surfaces, such as a co-rotating ceiling and showcase floors, is immaterial.

Additional assumptions apply to specific door types and are presented along with the derivations of the kinetic energy equations relating to those doors.

Basic Kinetic Energy Equations

In all instances, the rotational kinetic energy of a door component, or rigid grouping of components, is given by

$$E = \frac{1}{2} I \omega^2 \tag{1}$$

where E is the kinetic energy, I is the moment-of-inertia of the rotating component, or rigid grouping of components, and ω is the rate of rotation in radians per second.

The moment-of-inertia I in (1) expressed in terms of the parallel axis theorem is

$$I = M r^2 + I_{cm} \tag{2}$$

where M is the total mass of the component, or rigid grouping of components, r is the radial distance of the center-of-mass of the component, or rigid grouping of components, from the axis of rotation, and I_{cm} is the moment-of-inertia of the component, or rigid grouping of components, about its own center-of-mass.

When (2) expressing the parallel axis theorem is substituted into the basic rotational kinetic energy equation (1), the result is

$$E = \frac{1}{2} M r^2 \omega^2 + \frac{1}{2} I_{cm} \omega^2 . \tag{3}$$

Thus, the parallel axis theorem allows the basic rotational kinetic energy equation to be broken into the sum of the following two distinct parts.

$$E^{(orbit)} = \frac{1}{2} M r^2 \omega^2 \tag{4}$$

and

$$E^{(cm)} = \frac{1}{2} I_{cm} \omega^2 \ . \tag{5}$$

These two equations will serve as the starting point in every instance in the following in which the parallel axis theorem is invoked.

Kinetic Energy Equations of Basic 3-wing and 4-wing Doors

3-wing and 4-wing Doors without a Core

The following additional assumptions apply to the derivation of the rotational kinetic energy equations relating to 3-wing and 4-wing doors without a core.

1. The rotating part of the door consists of a number n of identical flat rectangular panels ganged rigidly together and oriented radially, where n is 3 or 4.

2. The panels revolve around a common vertical axis coincident with one vertical edge of each panel.

3. Other components of the door that may be in motion, such as a ceiling and the drive mechanism, are not considered here.

As a consequence of the assumptions listed on pages 4 and 7, the moment-of-inertia of each door panel is equivalent to that of a uniform bar rotating about one end. That is, to

$$I = \frac{1}{3} M L^2 , \tag{6}$$

where M is the mass of the bar and L is its length. Because of the equivalency just mentioned, M is here the mass of an individual door panel and L its width.

Since, in the case of 3-wing and 4-wing doors without a core, one vertical edge of each door panel is coincident with the axis of rotation, and since, by (6), the moment-of-inertia of each panel about its end and, therefore, about the axis of rotation is known, it is not necessary to invoke the parallel axis theorem. Rather, the moment-of-inertia (6) can be used directly in (1) to express the rotational kinetic energy of the i-th panel of the revolving door. The result is

$$E_i = \frac{1}{6} M L^2 \omega^2 , \tag{7}$$

where $i = 1, 2, 3... n$. Since the panels are assumed to be identical, the total kinetic energy of rotation of the door is simply n times that of an individual panel given by (7). That is, the total kinetic energy of rotation of the door is

$$E = \frac{1}{6} n M L^2 \omega^2 . \tag{8}$$

To express the kinetic energy in units of pound-feet, the units used by the ANSI A156.27 national standard for automatic revolving doors, mass M must be expressed in "equivalent" pounds and the width L of an individual door panel must be expressed in feet. Also, for compatibility with ANSI A156.27, the angular velocity ω must be expressed as the equivalent number of revolutions per second (rpm).

The pound is *not* a unit of mass, but a unit of force. But, with the understanding that weight in pounds, which is determined at sea level, stands in as a proxy for mass, the weight W in pounds that corresponds to the mass M (in slugs) of an individual door panel at sea level is given by Newton's first law, which relates force and mass, as

$$W = M g \quad or \quad M = \frac{W}{g}, \tag{9}$$

where g is the acceleration of gravity at sea level and has the approximate value 32.17 ft/sec^2.

Since one revolution is equal to 2π radians, and a minute contains 60 seconds,

$$\omega = \left(\frac{2\pi}{60}\right)\Omega = \frac{\pi}{30}\Omega, \tag{10}$$

where Ω is the rotation rate of the door in revolutions per minute (rpm).

Substitution of (9) and (10) into (8) gives,

$$E = \frac{1}{6}\left(\frac{n\,W}{g}\right)\left(\frac{\pi L}{30}\right)^2 \Omega^2 . \tag{11}$$

And, substituting the numerical values for g and π gives,

$$E = \frac{n W L^2 \Omega^2}{17601}, \tag{12}$$

where the total rotational kinetic energy E of the door is in units of pound-feet (lb-ft), n is the number of door panels, W is the weight of an individual door panel in pounds, L is the width of a door panel in feet, and Ω is the rotation rate of the door in revolutions per minute (rpm).

Kinetic Energy Equations for a 3-wing Door without a Core

Letting $n = 3$ for a 3-wing door without a core and expressing the wing width L in terms of the overall door diameter

$$L = \frac{D}{2}, \tag{13}$$

the rotational kinetic energy of a 3-wing door with no core is

$$\text{No core:} \qquad E_{3-wing} = \frac{W D^2 \Omega^2}{23468}, \tag{14}$$

where $E_{3\text{-}wing}$ is the rotational kinetic energy of the door in lb-ft, W is the weight of an individual door wing in pounds, D is the overall diameter of the door in feet and Ω is the rotation rate of the door in rpm.

Equation (14) can be used to calculate the rotation rate Ω of a 3-wing door without a core that results in the door carrying exactly 2.5 lb-ft of rotational kinetic energy by solving (14) for Ω with E set to 2.5 lb-ft. The result is,

$$\text{3-wing no core:} \qquad \Omega_{2.5} = \frac{242}{D\sqrt{W}}. \tag{15}$$

Likewise, with E set to 7.0 lb-ft, (14) can be solved for the rotation rate Ω of a 3-wing door without a core that results in the door carrying exactly 7.0 lb-ft of rotational kinetic energy. The result is,

$$\text{3-wing no core:} \qquad \Omega_{7.0} = \frac{405}{D\sqrt{W}}. \tag{16}$$

Kinetic Energy Equations for a 4-wing Door without a Core

And, setting $n = 4$ for a 4-wing door without a core and using (13) in (12) for the wing width L, the rotational kinetic energy of a 4-wing door with no core is

$$\text{No core:} \qquad E_{4-wing} = \frac{W D^2 \Omega^2}{17601}, \tag{17}$$

where $E_{4\text{-}wing}$ is the rotational kinetic energy of the door in lb-ft, W is the weight of an individual door wing in pounds, D is the overall diameter of the door in feet and Ω is the rotation rate of the door in rpm.

Equation (17) can be used to calculate the rotation rate Ω of a 4-wing door without a core that results in the door carrying exactly 2.5 lb-ft of rotational kinetic energy by solving (17) for Ω with E set to 2.5 lb-ft. The result is,

4-wing no core: $$\Omega_{2.5} = \frac{210}{D\sqrt{W}}\,. \qquad (18)$$

Likewise, with E set to 7.0 lb-ft, (17) can be solved for the rotation rate Ω of a 4-wing door without a core that results in the door carrying exactly 7.0 lb-ft of rotational kinetic energy. The result is,

4-wing no core: $$\Omega_{7.0} = \frac{351}{D\sqrt{W}}\,. \qquad (19)$$

3-wing and 4-wing Doors with a Core

The following additional assumptions apply to the derivation of the rotational kinetic energy equations relating to 3-wing and 4-wing doors with a core.

1. The rotating part of the door consists of a number n of identical flat rectangular radially directed panels each rigidly attached along one vertical edge to one vertex of an n-sided regular polygonal core.

2. The n sides of the regular polygonal core to which the radial panels are attached consist also of identical flat rectangular panels.

3. The core to which the panels are attached revolves around a vertical axis coincident with the geometric center of the polygonal core.

4. Other components of the door that may be in motion, such as a ceiling and the drive mechanism, are not considered here.

The Figure below illustrates the dimensions and naming conventions employed in the derivations using a 3-wing door as an example. However, because the equations for two door configurations are to be derived – those for a 3-wing and those for a 4-wing door with a core – the kinetic energy equations will first be derived for the general case of a door with n identical radial panels and an n-sided regular polygonal core. Then, n will be particularized to 3 and 4.

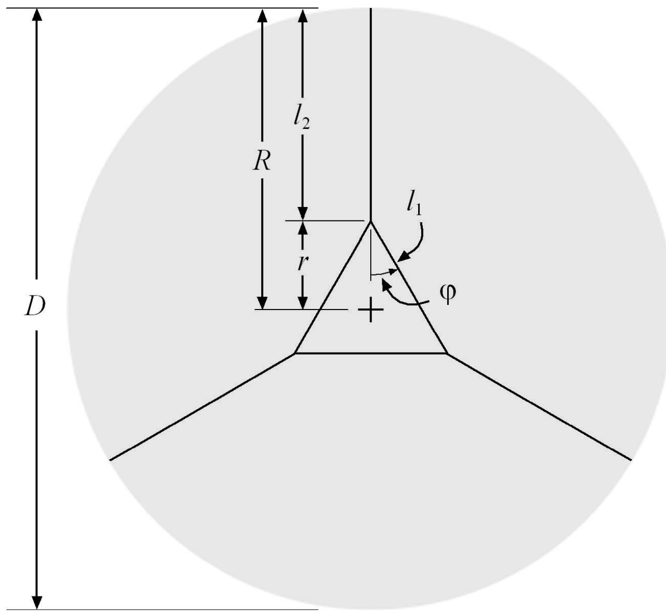

Figure 2. Dimensions and naming conventions for a revolving door with a regular polygonal core.

D represents the overall diameter of the door, R the radius of the door, r the radial distance from the geometric center of the door to a vertex of the polygonal core, l_1 the width of one side of the regular polygonal core, l_2 the width of each radial panel, and φ the internal half-angle subtended by a vertex of the regular polygonal core.

The derivation of the kinetic energy equations for the complete door with a core will start with the derivation of the kinetic energy contributed by a basic unit consisting of a radial panel and one of the core panels to which it is attached. The complete door comprises n such units so that the kinetic energy of the complete door will be n times the result derived for the basic unit. The basic unit is illustrated in the Figure below consisting of the radial panel l_2 and core panel l_1.

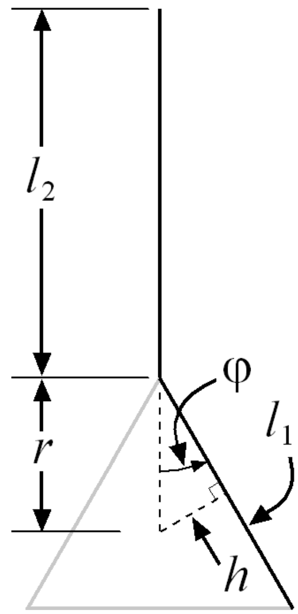

Figure 3. Basic unit of *n*-wing door with core.

h is the perpendicular distance from the geometric center of the regular polygonal core to the center of one side of the core.

The following relationships are clear from Figure 3.

$$h = r\sin\varphi \tag{20}$$

and

$$l_1 = 2r\cos\varphi\,. \tag{21}$$

It is also clear from Figure 2 that

$$R = \frac{D}{2} = l_2 + r\,, \quad \text{or} \quad r = \frac{D}{2} - l_2\,. \tag{22}$$

The interior angle subtended by the vertex of a regular plane polygon of *n* sides is

$$\pi\left(\frac{n-2}{n}\right).$$

Consequently,

$$\varphi = \frac{\pi}{2}\left(\frac{n-2}{n}\right). \tag{23}$$

As a consequence of the assumptions delineated above, the moment-of-inertia of each door panel, radial and core, about its own center-of-mass is equivalent to that of a uniform bar rotating about its center. That is, to

$$I = \frac{1}{12} M L^2 \tag{24}$$

where M is the mass of the bar and L is its length. Here, M is the mass of an individual door panel and L its width. Combining (5) and (24), the rotational kinetic energy of a panel of mass M and width L due to its rotation about its own center-of-mass is

$$E^{(cm)} = \frac{1}{24} M L^2 \omega^2 . \tag{25}$$

The kinetic energy due to the rotation of the center-of-mass of the panel about the axis of rotation of the door is from (4) with ρ substituted for r

$$E^{(orbit)} = \frac{1}{2} M \rho^2 \omega^2 \tag{26}$$

where ρ is the radial distance of the center-of-mass of the panel from the axis of rotation of the door.

Using the parallel axis theorem, the total kinetic energy of the panel due to its rigid rotation within the structure of the door is the sum of the two components given by (25) and (26). Namely,

$$E^{(panel)} = E^{(cm)} + E^{(orbit)} . \tag{27}$$

Let the mass of a core panel be m_1 and that of a radial panel m_2. From (25), the kinetic energy associated with the rotation of a core panel about its own center-of-mass is

$$E_1^{(cm)} = \frac{1}{24} m_1\, \omega^2\, l_1^2 = \frac{1}{6} m_1\, \omega^2\, r^2 \cos^2\varphi = \frac{1}{6} m_1\, \omega^2\, r^2 \left(1 - \sin^2\varphi\right), \tag{28}$$

where (21) for l_1 has also been introduced. And, from (26), the kinetic energy associated with the rotation of the center-of-mass of a core panel about the axis of rotation of the door is

$$E_1^{(orbit)} = \frac{1}{2} m_1\, \omega^2\, h^2 = \frac{1}{2} m_1\, \omega^2 r^2 \sin^2\varphi\,, \tag{29}$$

where h has been substituted for ρ and (20) for h has been introduced. Consequently, the total kinetic energy of a core panel is, from (27), (28) and (29),

$$E_1^{(panel)} = \frac{1}{2} m_1\, \omega^2\, r^2 \left(\sin^2\varphi + \frac{1 - \sin^2\varphi}{3} \right) = \frac{1}{6} m_1\, \omega^2\, r^2 \left(1 + 2\sin^2\varphi\right). \tag{30}$$

And, with the substitution of (22) for r in (30),

$$E_1^{(panel)} = \frac{1}{6} m_1\, \omega^2 \left(\frac{D}{2} - l_2 \right)^2 \left(1 + 2\sin^2\varphi\right). \tag{31}$$

In the same manner, the kinetic energy associated with the rotation of a radial panel about its own center-of-mass is from (25)

$$E_2^{(cm)} = \frac{1}{24} m_2\, \omega^2\, l_2^2\,. \tag{32}$$

Since the mass distribution of the radial panel is uniform across the width of the panel, the center-of-mass of the radial panel is located at a distance $l_2/2$ from either edge (side) of the panel. Referring to either Figure 2 or 3, it is clear then that the radial distance of the center-of-mass of a radial panel from the axis of rotation of the door is

$$\rho = r + \frac{l_2}{2} = \left(\frac{D}{2} - l_2 \right) + l_2 = \frac{D - l_2}{2}\,, \tag{33}$$

where (22) for r has been introduced.

The kinetic energy associated with the rotation of the center-of-mass of a radial panel about the axis of rotation of the door is found from (26) and (33) to be

$$E_2^{(orbit)} = \frac{1}{2} m_2 \omega^2 \rho^2 = \frac{1}{8} m_2 \omega^2 \left(D - l_2\right)^2 . \tag{34}$$

The kinetic energy of the combination of a core panel and a radial panel, the basic unit considered here, is

$$E^{(unit)} = E_1^{(cm)} + E_1^{(orbit)} + E_2^{(cm)} + E_2^{(orbit)} = E_1^{(panel)} + E_2^{(cm)} + E_2^{(orbit)} ,$$

which, from (31), (32) and (34), is found to be

$$E^{(unit)} = \frac{1}{6} m_1 \omega^2 \left(\frac{D}{2} - l_2\right)^2 \left(1 + 2\sin^2\varphi\right) + \frac{1}{8} m_2 \omega^2 \left[\left(D - l_2\right)^2 + \frac{l_2^2}{3}\right]. \tag{35}$$

Clearing the fractions from the factors in parentheses and brackets and taking ω to the left gives

$$E^{(unit)} = \frac{\omega^2}{24} \left\{ m_1 \left(D - 2l_2\right)^2 \left(1 + 2\sin^2\varphi\right) + m_2 \left[3\left(D - l_2\right)^2 + l_2^2\right]\right\}. \tag{36}$$

And expanding the factors involving D gives

$$E^{(unit)} = \frac{\omega^2}{24} \left\{ m_1 \left(D^2 - 4Dl_2 + 4l_2^2\right)\left(1 + 2\sin^2\varphi\right) + m_2 \left(3D^2 - 6Dl_2 + 4l_2^2\right)\right\} \tag{37}$$

Result (37) can now be particularized for a 3-wing or 4-wing door by setting n equal to 3 or 4, respectively. With reference to (23),

$$\sin\varphi = \sin\frac{\pi}{6} = \frac{1}{2}, \qquad (n = 3) \tag{38a}$$

$$\sin\varphi = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \qquad (n = 4) \tag{38b}$$

so that the trigonometric factor in the first term of (37) becomes

$$1+2\sin^2\varphi = 1+2\left(\frac{1}{2}\right)^2 = \frac{3}{2} \qquad (n=3) \tag{39a}$$

$$1+2\sin^2\varphi = 1+2\left(\frac{1}{\sqrt{2}}\right)^2 = 2 \qquad (n=4) \tag{39b}$$

Kinetic Energy Equations for a 3-wing Door with a Polygonal Core

A 3-wing door with a polygonal core is comprised of n = 3 units of the type depicted in Figure 3. Thus,

$$E_{3-wing} = nE^{(unit)} = 3E^{(unit)}, \tag{40}$$

where (37) for $E^{(unit)}$ is to be evaluated for the case n = 3 using (39a). Thus, (37), (39a) and (40) yield

$$E_{3-wing} = \frac{\omega^2}{8}\left[\frac{3}{2}m_1\left(D^2 - 4Dl_2 + 4l_2^2\right) + m_2\left(3D^2 - 6Dl_2 + 4l_2^2\right)\right]. \tag{41}$$

Grouping by descending powers of D yields

$$E_{3-wing} = \frac{\omega^2}{8}\left[D^2\left(\frac{3}{2}m_1 + 3m_2\right) - 6Dl_2\left(m_1 + m_2\right) + l_2^2\left(6m_1 + 4m_2\right)\right]. \tag{42}$$

And taking a factor of 6 out of the brackets gives

$$E_{3-wing} = \frac{3}{4}\omega^2\left[\left(\frac{D}{2}\right)^2\left(m_1 + 2m_2\right) - Dl_2\left(m_1 + m_2\right) + l_2^2\left(m_1 + \frac{2}{3}m_2\right)\right]. \tag{43}$$

To express this kinetic energy in units of pound-feet, the units used by the ANSI A156.27 national standard for automatic revolving doors, the masses m_1 and m_2 must be expressed in "equivalent" pounds and the width of the radial panel l_2 and the overall door diameter D expressed in feet. Also, the angular velocity ω must be expressed as equivalent revolutions per second for compatibility with ANSI A156.27.

Substituting (9) and (10) into (43) and using the numerical values for g and π gives,

$$E_{3-wing} = \frac{1}{3911}\left[\left(\frac{D}{2}\right)^2 (w_1 + 2w_2) - Dl_2(w_1 + w_2) + l_2^2\left(w_1 + \frac{2}{3}w_2\right)\right]\Omega^2 , \qquad (44)$$

where the total rotational kinetic energy E_{3-wing} of the door is in units of pound-feet (lb-ft), w_1 and w_2 are the weights of a core panel and a radial panel, respectively, in pounds, l_2 is the width of a radial panel and D the overall door diameter both in feet, and Ω is the rotation rate of the door in revolutions per minute (rpm).

Equation (44) can be used to calculate the rotation rate Ω of the door that results in the door carrying exactly 2.5 lb-ft of rotational kinetic energy by solving (44) for Ω with E_{3-wing} set to 2.5 lb-ft. The result is,

$$\Omega_{2.5} = \frac{98.9}{\sqrt{\left(\frac{D}{2}\right)^2 (w_1 + 2w_2) - Dl_2(w_1 + w_2) + l_2^2\left(w_1 + \frac{2}{3}w_2\right)}} . \qquad (45)$$

Equation (44) may likewise be used to determine the rotation rate Ω of the door that results in the door carrying exactly 7.0 lb-ft of rotational kinetic energy by solving (44) for Ω with E_{3-wing} set to 7.0 lb-ft. The result is,

$$\Omega_{7.0} = \frac{165}{\sqrt{\left(\frac{D}{2}\right)^2 (w_1 + 2w_2) - Dl_2(w_1 + w_2) + l_2^2\left(w_1 + \frac{2}{3}w_2\right)}} . \qquad (46)$$

Kinetic Energy Equations for a 4-wing Door with a Polygonal Core

A 4-wing door with a polygonal core is comprised of n = 4 units of the type depicted in Figure 3. Thus,

$$E_{4-wing} = nE^{(unit)} = 4E^{(unit)} , \qquad (47)$$

where (37) for $E^{(unit)}$ is now to be evaluated for the case n = 4 using (39b). Thus, (37), (39b) and (47) yield

$$E_{4-wing} = \frac{\omega^2}{6}\left[2m_1\left(D^2 - 4Dl_2 + 4l_2^2\right) + m_2\left(3D^2 - 6Dl_2 + 4l_2^2\right)\right]. \qquad (48)$$

Grouping by descending powers of *D* yields

$$E_{4-wing} = \frac{\omega^2}{6}\left[D^2\left(2m_1 + 3m_2\right) - 2Dl_2\left(4m_1 + 3m_2\right) + 4l_2^2\left(2m_1 + m_2\right)\right]. \qquad (49)$$

And taking a factor of 2 out of the brackets gives

$$E_{4-wing} = \frac{\omega^2}{3}\left[\frac{D^2}{2}\left(2m_1 + 3m_2\right) - Dl_2\left(4m_1 + 3m_2\right) + 2l_2^2\left(2m_1 + m_2\right)\right]. \qquad (50)$$

Substituting (9) and (10) into (50) and using the numerical values for *g* and *π* gives,

$$E_{4-wing} = \frac{1}{8801}\left[\frac{D^2}{2}\left(2w_1 + 3w_2\right) - Dl_2\left(4w_1 + 3w_2\right) + 2l_2^2\left(2w_1 + w_2\right)\right]\Omega^2. \qquad (51)$$

where the total rotational kinetic energy E_{4-wing} of the door is in units of pound-feet (lb-ft), w_1 and w_2 are the weights of a core panel and a radial panel, respectively, in pounds, l_2 is the width of a radial panel and *D* the overall door diameter both in feet, and Ω is the rotation rate of the door in revolutions per minute (rpm).

Equation (51) can be used to calculate the rotation rate Ω of the door that results in the door carrying exactly 2.5 lb-ft of rotational kinetic energy by solving (51) for Ω with E_{4-wing} set to 2.5 lb-ft. The result is,

$$\Omega_{2.5} = \frac{148}{\sqrt{\frac{D^2}{2}\left(2w_1 + 3w_2\right) - Dl_2\left(4w_1 + 3w_2\right) + 2l_2^2\left(2w_1 + w_2\right)}}. \qquad (52)$$

Equation (51) may likewise be used to determine the rotation rate Ω of the door that results in the door carrying exactly 7.0 lb-ft of rotational kinetic energy by solving (51) for Ω with E_{4-wing} set to 7.0 lb-ft. The result is,

$$\Omega_{7.0} = \frac{248}{\sqrt{\frac{D^2}{2}\left(2w_1 + 3w_2\right) - Dl_2\left(4w_1 + 3w_2\right) + 2\,l_2^2\left(2w_1 + w_2\right)}}\ . \tag{53}$$

Kinetic Energy Equations for 2-wing Doors

The rotational kinetic energy equations for a 2-wing revolving door equipped with a showcase and concentric curved panel attached to the outer end of each wing are now derived. The following additional assumptions are made.

1. The rotating elements of the door correspond to those illustrated in the plan view shown in Figure 4 and are assumed to rotate together as a rigid unit.

2. The mass of each flat panel is distributed uniformly in the horizontal direction across the width of the panel. This assumption is not necessary for the curved panels because all points within each curved panel lie at the same distance R from the axis of rotation.

3. Other components of the door that may be in motion, such as the ceiling, showcase floor and the drive mechanism, are not considered here.

Since one vertical edge of each radial panel, of width r and identified by the subscript p in Figure 4, is coincident with the axis of rotation, and since, by (6), the moment-of-inertia of each panel about its end and, therefore, about the axis of rotation is known, it is not necessary to invoke the parallel axis theorem for the radial panels. Rather, the moment-of-inertia (6) can be used directly to express the rotational kinetic energy of the radial panels.

By (6), the moment-of-inertia about the axis of rotation of each radial panel is

$$I_p = \frac{1}{3} m_p r^2 = \frac{1}{3} \frac{w_p}{g} r^2 \quad , \tag{54}$$

where I_p is the moment-of-inertia about the axis of rotation, m_p is the mass of the panel, w_p is the weight of the panel, r is the width of the panel and g is the acceleration of gravity at sea level. The approximate value of g is 32.17 ft/sec^2. Newton's first law relating force, mass and acceleration has been used in (54) to express the mass of the radial panel m_p in terms of the weight of the panel w_p at sea level.

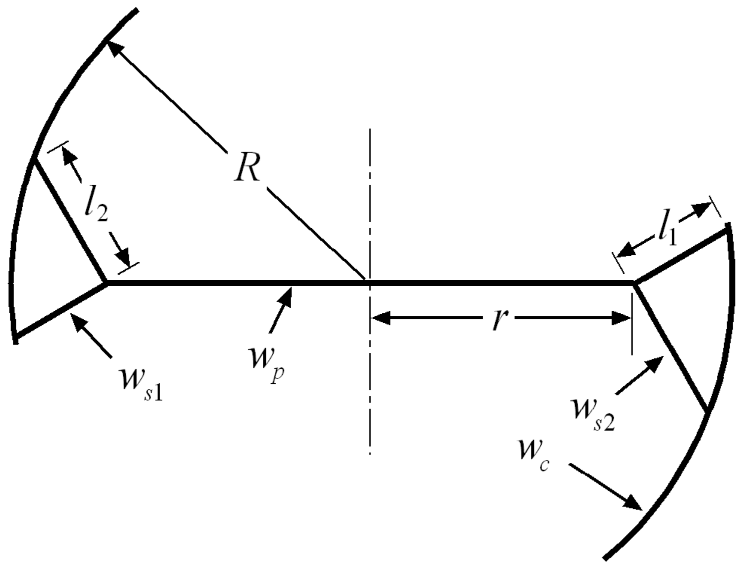

Figure 4. Plan view of the revolving elements of the 2-wing revolving door.

Since the elements of mass of the curved panels identified by the subscript c in Figure 4 all lie at the same distance R from the axis of rotation, the moment-of-inertia of each curved panel about the axis of rotation is simply

$$I_c = m_c R^2 = \frac{w_c}{g} R^2 \,, \tag{55}$$

where again Newton's law has been used to express the mass of the curved panel in terms of its weight at sea level.

The moments-of-inertia about the axis of rotation at the geometric center of the door of the remaining flat showcase panels of widths l_1 and l_2 and identified by the subscripts $s1$ and $s2$ in Figure 4, are derived using the parallel axis theorem.

The moment-of-inertia about the center-of-mass of each showcase panel is given by the formula for the moment-of-inertia about its center of a bar of uniform mass

$$I = \frac{1}{12} m l^2 , \qquad (56)$$

where *m* is the mass of the bar and *l* the length of the bar. In this application, *m* is the mass of the respective showcase panel and *l* is its width.

To calculate the moments-of-inertia of the centers-of-mass of the flat showcase panels about the axis of rotation, it is necessary first to have an expression for the distance of the center-of-mass of each showcase panel from the axis of rotation. Because the masses of the showcase panels are assumed to be uniform in the horizontal direction, the center-of-mass of each such panel is located midway along the panel. One end of each panel lies at a distance *r* from the axis of rotation, and the other end at a distance *R*. If the width of the panel is *l*, it is necessary, therefore, to develop an expression for the distance ρ illustrated in Figure 5.

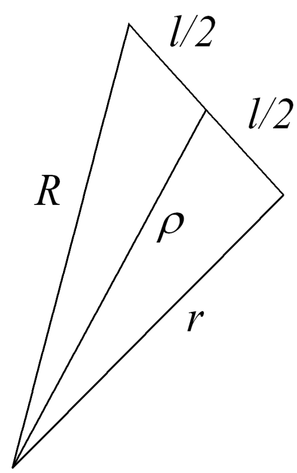

Figure 5. Distance ρ to center of mass of showcase panel.

Treating *R*, *r*, ρ and *l* as vectors, the perimeter triangle in Figure 5 satisfies the relationship

$$r^2 = R^2 + l^2 - 2R \cdot l$$

where "·" represents the vector inner, or "dot", product and the square of a vector represents the square of the magnitude of the vector. Likewise, the upper triangle in Figure 5 satisfies the relationship

$$\rho^2 = R^2 + \frac{l^2}{4} - R \cdot l \,.$$

When the first of these relationships is used to eliminate $R \cdot l$ from the second, the result is

$$\rho^2 = \frac{1}{2}\left(r^2 + R^2\right) - \frac{l^2}{4} \,. \qquad (57)$$

Using (56) and (57), the total moment-of-inertia of the first showcase panel with respect to the axis of rotation at the center of the door is, from the parallel axis theorem (2),

$$I_1 = \frac{1}{12} m_{s1} l_1^2 + m_{s1}\left[\frac{1}{2}\left(r^2 + R^2\right) - \frac{l_1^2}{4}\right],$$

or

$$I_1 = \frac{1}{2}\frac{w_{s1}}{g}\left(r^2 + R^2\right) - \frac{1}{6}\frac{w_{s1}}{g} l_1^2 \,. \qquad (58)$$

where Newton's law has again been used to express the mass of the panel in terms of its weight at sea level.

In exactly the same way, the total moment-of-inertia of the second showcase panel about the axis of rotation at the geometric center of the door is

$$I_2 = \frac{1}{2}\frac{w_{s2}}{g}\left(r^2 + R^2\right) - \frac{1}{6}\frac{w_{s2}}{g} l_2^2 \,. \qquad (59)$$

The total moment-of-inertia of all rotating components of the door illustrated in Figure 4, taking into account also that there are two identical sections to the door, is twice the sum of (54), (55), (58) and (59). That is

$$I = \frac{2}{g}\left[w_c R^2 + \frac{w_{s1} + w_{s2}}{2}\left(r^2 + R^2\right) - \frac{1}{6}\left(w_{s1} l_1^2 + w_{s2} l_2^2\right) + \frac{1}{3} w_p r^2\right]. \qquad (60)$$

The expression (1) for the rotational kinetic energy of the door can be expressed in terms of the rotation rate of the door in revolutions per minute (rpm) by substituting (10) into (1) to get

$$E = \frac{1}{2} I \left(\frac{\pi}{30} \right)^2 \Omega^2 . \tag{61}$$

where Ω is the rotation rate of the door in revolutions per minute.

When (60) is substituted for *I* in (61) and the numerical values of *π* and *g* utilized, the final result for the rotational kinetic energy of the 2-wing revolving door is

$$E = \frac{1}{2933.6} \left[w_c R^2 + \frac{w_{s1} + w_{s2}}{2} \left(r^2 + R^2 \right) - \frac{1}{6} \left(w_{s1} l_1^2 + w_{s2} l_2^2 \right) + \frac{1}{3} w_p r^2 \right] \Omega^2 \tag{62}$$

where the rotational kinetic energy *E* is in units of pound-feet; the weights w_c, w_{s1}, w_{s2} and w_p of the door components are in pounds; the dimensions *r*, *R*, l_1 and l_2 are in feet; and the rotation rate of the door Ω is in revolutions per minute.

Kinetic Energy Equations for 2-wing Doors with Showcases

With the substitution

$$R = \frac{D}{2},$$

equation (62) can be written equivalently as

$$E_{2-wing} = \frac{1}{17601} \left[\frac{3D^2}{4} \left(2w_c + w_{s1} + w_{s2} \right) + r^2 \left[3\left(w_{s1} + w_{s2} \right) + 2w_p \right] - \left(w_{s1} l_1^2 + w_{s2} l_2^2 \right) \right] \Omega^2 \tag{63}$$

where again the rotational kinetic energy $E_{2\text{-}wing}$ is in units of pound-feet; the weights w_c, w_{s1}, w_{s2} and w_p of the door components are in pounds; the dimensions *r*, *D*, l_1 and l_2 are in feet; and the rotation rate of the door Ω is in revolutions per minute.

Equation (63) can be solved for the rotation rate of the door in revolutions per minute Ω at which the door exhibits 2.5 lb-ft and 7.0 lb-ft of rotational kinetic energy. The results are

$$\Omega_{2.5} = \frac{210}{\sqrt{\frac{3D^2}{4}\left(2w_c + w_{s1} + w_{s2}\right) + r^2\left[3\left(w_{s1} + w_{s2}\right) + 2\,w_p\right] - \left(w_{s1}\,l_1^2 + w_{s2}\,l_2^2\right)}} \tag{64}$$

and,

$$\Omega_{7.0} = \frac{351}{\sqrt{\frac{3D^2}{4}\left(2w_c + w_{s1} + w_{s2}\right) + r^2\left[3\left(w_{s1} + w_{s2}\right) + 2\,w_p\right] - \left(w_{s1}\,l_1^2 + w_{s2}\,l_2^2\right)}} \tag{65}$$

Kinetic Energy Equations for Showcase Floors

The kinetic energy equations derived to this point have been simple enough that they can be expressed for each door type in a single, closed-form expression. However, this is not the case with the derivation of the kinetic energy equations for the showcase floors associated with a 2-wing revolving door. Rather, the derivation will result instead in a series of mathematical steps – an algorithm – for the computation of the kinetic energy.

There are two basic reasons why the equations for the showcase floors are so much more complicated than those considered thus far. The first is that the shape of the showcase floor is not a simple straight line or curve of constant radius, as are the flat and curved panels in the previous derivations. Instead, the showcase floor has a roughly triangular shape, being bounded on two sides by flat showcase panels and on the third side by a curved panel of constant radius. The self moment-of-inertia and center-of-mass location of this shape, required for application of the parallel axis theorem, are not readily available in the handbooks and other sources, thereby immediately introducing the added complication that these parameters must be derived *ab initio*.

The second reason is that the showcase floor becomes, in general, a different shape if its relative dimensions change. For example, a fundamentally new shape arises if the width of one of the two flat showcase panels changes while the other panel sizes remain fixed. The new shape cannot be made congruent with the original shape by merely changing the overall scale. This is not true of the straight and curved lines considered thus far. The result is that the kinetic energy equations for the showcase floor are necessarily more complicated because they must encode the infinity of unique shapes that the floor could have.

Method

In order to overcome the additional complexity imposed by the shape of the showcase floor, the kinetic energy derivation will be broken into the sum of three parts, each of which involves a shape that is much more easily analyzed.

The shaded area in Figure 6 below illustrates the general shape of a conventional showcase floor. It is bounded on two sides by flat showcase panels and on the third, peripheral side by a curved panel of constant radius. To simplify the analysis, the floor will be considered as the sum of three parts – the two right triangles labeled "triangle l" and "triangle 2" plus the segment of the circle bounded by the curved panel on the right and, on the left, by the chord between the points at which the flat showcase panels

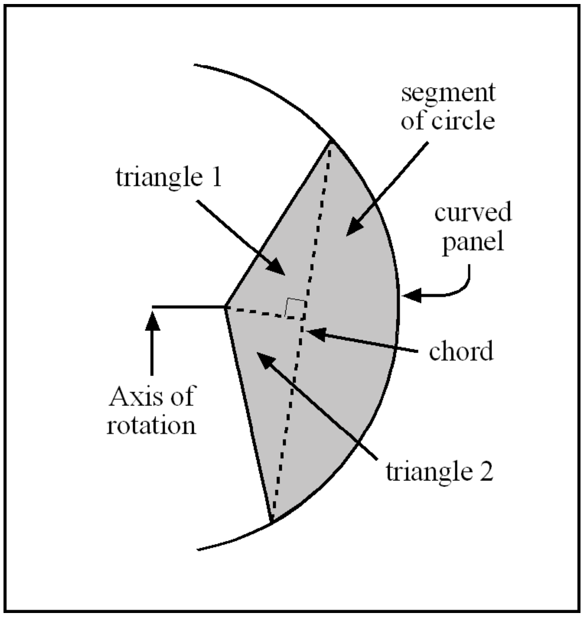

Figure 6. Showcase floor, conventional shape.

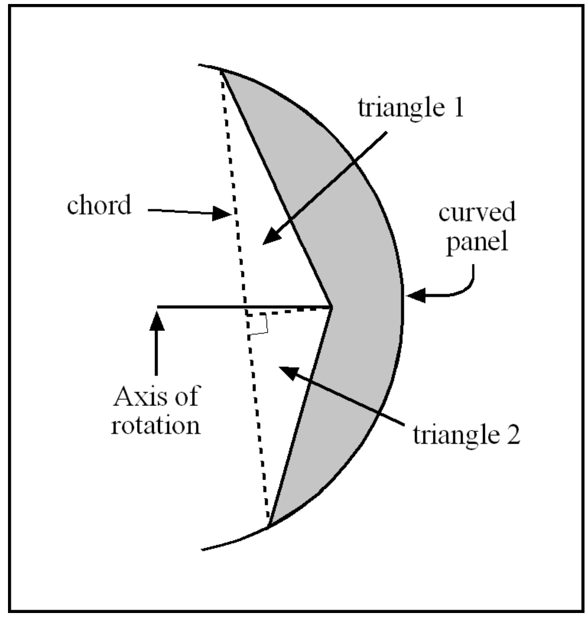

Figure 7. Showcase floor, non-conventional shape.

meet the curved panel. The self moment-of-inertia of a right triangle and the known location of the center-of-mass of a right triangle will be used, along with the parallel axis theorem, to derive the contributions to the kinetic energy from the two right triangles comprising part of the showcase floor. On the other hand, due to its geometry and curvature centered on the axis of rotation, the kinetic energy contributed by the segment of a circle will be derived by direct integration, with no need to invoke the parallel axis theorem. The kinetic energy of the complete showcase floor is equal to the sum of the kinetic energies of these three parts.

Although the non-conventional showcase floor shape illustrated in Figure 7 above may be rarely, if ever, encountered, the RDEC kinetic energy equations for showcase floors have been generalized to handle this case as well. The method is essentially the same as that just described for the conventional shape illustrated in Figure 6 except that the contributions to the kinetic energy of the showcase floor from the two right triangles are subtracted from, rather than added to, the contribution from the segment of a circle. As can be seen in Figure 7, the shaded area representing the showcase floor is equal to the area of the segment of a circle bounded on the left by the chord, ***minus*** the areas of the two right triangles. The ability to add or subtract, as appropriate, the contributions from the right triangles is all that is required for RDEC to be able to cover both cases illustrated in Figures 6 and 7.

Algorithm

Figure 8 below displays the parameters that enter into the derivation of the kinetic energy equations for a showcase floor. Not defined in Figure 8, but also required, are the weight W in pounds of a single showcase floor and the rotation rate of the door ω in radians per second. Right triangle 1 has l_1 as its hypotenuse, right triangle 2 has l_2 as its

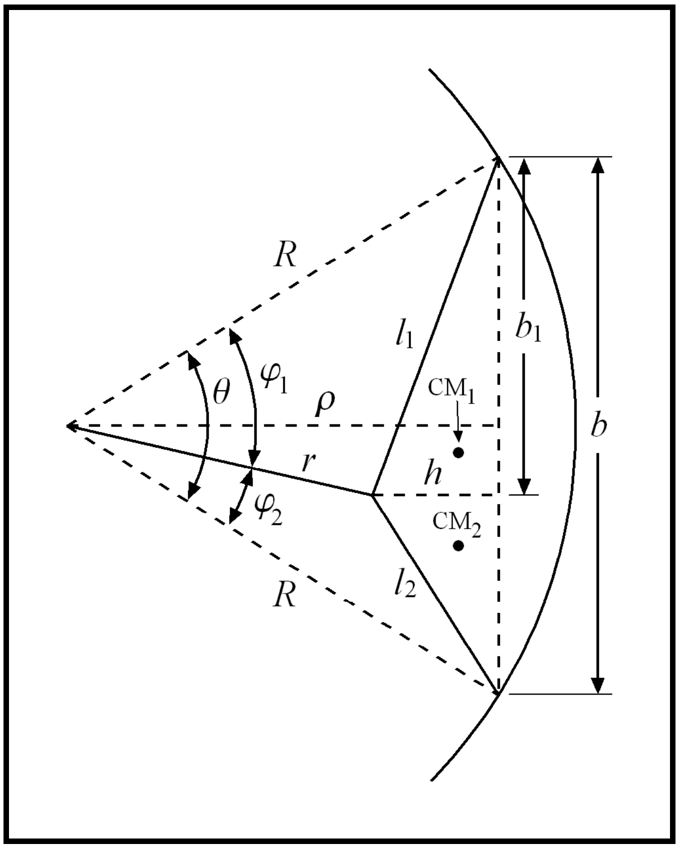

Figure 8. Diagram illustrating parameters used in the showcase kinetic energy equation derivations.

hypotenuse and the segment of the circle has chord b. ρ is the perpendicular bisector of the chord b extending from the axis of rotation. h is the common height of the two right triangles 1 and 2, and is also perpendicular to the chord b. CM_1 and CM_2 represent the locations of the centers-of-mass of triangles 1 and 2, though they are not necessarily positioned to scale in the diagram in Figure 8.

The first step is to determine the angles about the axis of rotation φ_1 and φ_2 subtended by the two flat showcase panels l_1 and l_2, respectively. These are found from the generalization of the Pythagorean theorem to the triangles of sides R, r and l_1, and of R, r and l_2. Namely

$$l_1^2 = R^2 + r^2 - 2Rr\cos\varphi_1 \quad \text{and} \quad l_2^2 = R^2 + r^2 - 2Rr\cos\varphi_2$$

from which it follows that

$$\varphi_1 = \cos^{-1}\left(\frac{R^2 + r^2 - l_1^2}{2Rr}\right) \tag{66a}$$

and,

$$\varphi_2 = \cos^{-1}\left(\frac{R^2 + r^2 - l_2^2}{2Rr}\right) \tag{66b}$$

From Figure 8 it is clear that the angle θ subtended by the chord b is

$$\theta = \varphi_1 + \varphi_2\,. \tag{67}$$

Noting that ρ is the perpendicular bisector of chord b, it can be seen immediately from Figure 8 and is also well known from mensuration that the length of the chord b is

$$b = 2R\sin\frac{\theta}{2}\,. \tag{68}$$

Likewise, ρ can be found from inspection of Figure 8, and is

$$\rho = R\cos\frac{\theta}{2}\,. \tag{69}$$

Figure 9 below illustrates the situation in which, as described under Method on page 26, the triangles 1 and 2 subtract from, rather than add to, the area of the segment of a circle. By comparing Figures 8 and 9, it is clear that the projection of r in the direction of the perpendicular bisector ρ of the chord b is less than ρ when the triangles add to, and greater than ρ when the triangles subtract from, the segment of a circle. The projection of r in the direction of ρ is from either Figure 8 or 9

$$proj = r\cos\left(\varphi_1 - \frac{\theta}{2}\right) = r\cos\left(\frac{\varphi_1 - \varphi_2}{2}\right) \tag{70}$$

where (67) has been substituted for θ in (70). Thus, to flag the addition or subtraction of the areas and kinetic energies of triangles 1 and 2, a variable s is set according to

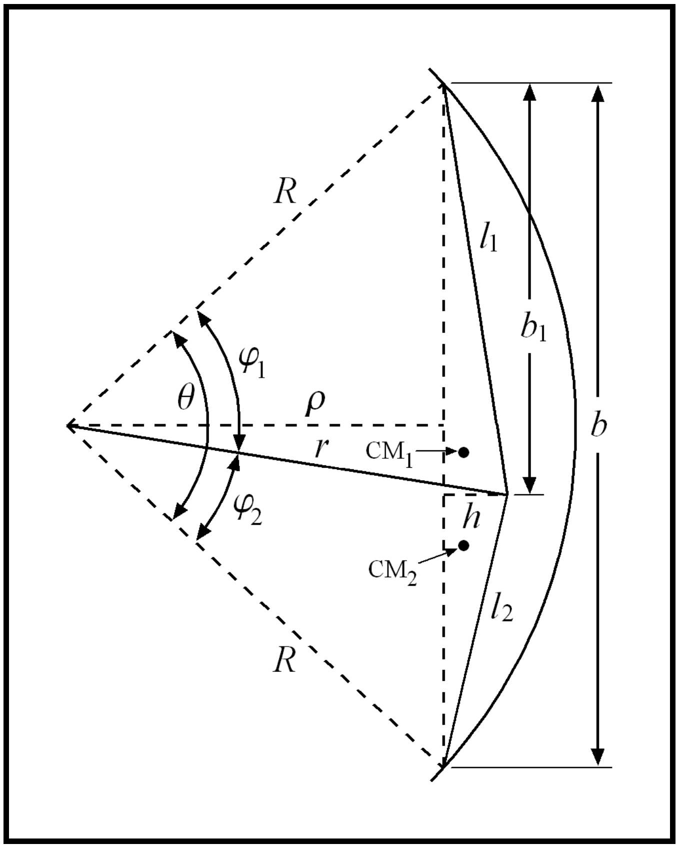

Figure 9. Illustrating the situation in which triangles 1 and 2 subtract from the area of the segment of the circle.

$$\text{IF } (proj > \rho),\ s = -1 \tag{71a}$$

$$\text{IF } (proj \leq \rho),\ s = +1 \tag{71b}$$

When $s = +1$, the effects of triangles 1 and 2 are to be added to those of the segment of a circle. And, when $s = -1$, the effects of the triangles are to be subtracted from the segment of a circle. When $proj = \rho$, the common height h of triangles 1 and 2 is zero so that, whether added to or subtracted from the segment of a circle, the triangles in this case have no effect on the kinetic energy.

From either Figure 8 or 9, the Pythagorean theorem applied to right triangles 1 and 2 gives

$$l_1^2 = b_1^2 + h^2 \tag{72}$$

and

$$l_2^2 = \left(b - b_1\right)^2 + h^2 = b^2 - 2bb_1 + b_1^2 + h^2 \,. \tag{73}$$

The length of the side b_1 of triangle 1 coincident with the chord b can be found by subtracting (73) from (72) and solving for b_1 to get

$$b_1 = \frac{l_1^2 - l_2^2 + b^2}{2b} \,. \tag{74}$$

And, the common height h of triangles 1 and 2 in terms of l_1 and b_1 is found from (72) to be

$$h = \sqrt{l_1^2 - b_1^2} \,. \tag{75}$$

In order to apply the parallel axis theorem to the right triangles 1 and 2, it is necessary to compute the radial distances from the axis of rotation of the centers-of-mass of the two triangles. As shown in Appendix 1, the center-of-mass of a right triangle with perpendicular sides a and b is located at a distance $a/3$ from the right angle along the side of length a and at a distance $b/3$ from the right angle along the side of length b.

It is easily deduced from either Figure 8 or 9 and the result from Appendix 1 that, since ρ divides the chord b into two equal parts of length $b/2$, the magnitude of the displacement of the center-of-mass of triangle 1 normal to ρ is

$$\left| \frac{b}{2} - \frac{2}{3} b_1 \right| . \tag{76}$$

And, clearly, the displacement of the center-of-mass of right triangle 1 along ρ is plus-or-minus $h/3$ depending upon whether the contribution of triangle 1 is to be added to or subtracted from the contribution of the segment of a circle. That is, upon the variable s defined by (71). If s is +1, then $h/3$ must be subtracted from ρ. Otherwise, if s is −1, $h/3$ must be added to ρ.

Using the Pythagorean theorem and the normal and parallel displacements from the point at which ρ intersects the chord b, the squared magnitude of the radial distance ρ_1 of the center-of-mass of triangle 1 from the axis of rotation is

$$\rho_1^2 = \left(\rho - \frac{sh}{3}\right)^2 + \left(\frac{b}{2} - \frac{2}{3}b_1\right)^2. \qquad (77)$$

It is likewise easily deduced from either Figure 8 or 9 and the result from Appendix 1 that, since ρ divides the chord b into two equal parts of length $b/2$, the magnitude of the displacement of the center-of-mass of triangle 2 normal to ρ is

$$\left|\frac{b}{6} - \frac{2}{3}b_1\right|. \qquad (78)$$

As before, the magnitude of the displacement of the center-of-mass of triangle 2 along ρ is $h/3$ with variable s determining whether $h/3$ is to be added to or subtracted from ρ.

Again using the Pythagorean theorem and the normal and parallel displacements from the point at which ρ intersects the chord b, the squared magnitude of the radial distance ρ_2 of the center-of-mass of triangle 2 from the axis of rotation is

$$\rho_2^2 = \left(\rho - \frac{sh}{3}\right)^2 + \left(\frac{b}{6} - \frac{2}{3}b_1\right)^2. \qquad (79)$$

The next step is to compute the areas of triangles 1 and 2 and the segment of the circle. The areas of the triangles are

$$A_1 = \frac{1}{2}sb_1h \qquad (80a)$$

and

$$A_2 = \frac{1}{2}s\left(b - b_1\right)h \qquad (80b)$$

where the factor of s has been included in the equations for the areas in order subsequently to cause the contributions of the triangles either to add to or subtract from the contribution from the segment of a circle.

The area of the segment of a circle is

$$A_s = \frac{R^2}{2}\left(\theta - \sin\theta\right). \tag{81}$$

This relationship is well known from mensuration and can be found in virtually any mathematical handbook.

Assuming that all length measurements are in feet, the area mass density λ of the showcase floor in slugs per square foot can be calculated from the weight W of the showcase floor, the areas (80a), (80b) and (81) and the acceleration of gravity at sea level g = 32.17 ft/sec^2 as

$$\lambda = \frac{1}{g}\frac{W}{A_1 + A_2 + A_s} \tag{82}$$

where (9) has been used to relate mass in slugs to weight in pounds. Given the sign attached to the areas A_1 and A_2 by s in (80a) and (80b), (82) correctly accounts for the actual area of the showcase floor indicated by the grey areas in Figures 6 and 7.

As shown in Appendix 2, the moment-of-inertia of a right triangle with perpendicular sides of length a and b about its own center-of-mass is

$$I_{cm} = \frac{M}{18}\left(a^2 + b^2\right). \tag{83}$$

where M is the mass of the triangle.

The mass M_1 of triangle 1 is $\lambda\, A_1$ and the lengths of its perpendicular sides are h and b_1. Therefore, the kinetic energy contributed by the rotation of triangle 1 about its own center-of-mass is, using (5)

$$E_1^{(cm)} = \frac{\lambda A_1}{36}\left(h^2 + b_1^2\right)\omega^2 = \frac{\lambda A_1}{36} l_1^2 \omega^2 . \tag{84}$$

And, the kinetic energy contributed by rotation of the center-of-mass of triangle 1 about the axis of rotation is from (4)

$$E_1^{(orbit)} = \frac{1}{2} M_1 \rho_1^2\, \omega^2 = \frac{1}{2}\lambda A_1 \rho_1^2 \omega^2 . \tag{85}$$

The total contribution to the kinetic energy from triangle 1 is the sum of (84) and (85),

$$E_1 = \frac{\lambda A_1}{2}\left(\rho_1^2 + \frac{l_1^2}{18}\right)\omega^2 = \frac{\lambda A_1}{182.38}\left(\rho_1^2 + \frac{l_1^2}{18}\right)\Omega^2 , \qquad (86)$$

where in the last step (10) has been used to relate ω in radians per second to Ω in rpm and the numerical value of π has been applied. Again, the sign of E_1 determined by the sign of A_1 will cause the kinetic energy E_1 due to triangle 1 properly to be added to or subtracted from the kinetic energy of the segment of a circle. With all lengths measured in feet and the mass of the showcase floor expressed in "equivalent" pounds, E_1 from (86) is in units of pound-feet (lb-ft).

Carrying through in the same way for triangle 2, but with area A_2, radial distance ρ_2 and perpendicular sides of length h and $b - b_1$, the contribution to the kinetic energy E_2 by triangle 2 is

$$E_2 = \frac{\lambda A_2}{2}\left(\rho_2^2 + \frac{l_2^2}{18}\right)\omega^2 = \frac{\lambda A_2}{182.38}\left(\rho_2^2 + \frac{l_2^2}{18}\right)\Omega^2 \qquad (87)$$

where again (10) has been used in the last step to relate ω in radians per second to Ω in rpm and the numerical value of π has been applied. And as before, the sign of E_2 determined by the sign of A_2 will cause the kinetic energy E_2 due to triangle 2 properly to be added to or subtracted from the kinetic energy of the segment of a circle. With all lengths measured in feet and the mass of the showcase floor expressed in "equivalent" pounds, E_2 from (87) is in units of lb-ft.

As derived in Appendix 3, the moment-of-inertia of the segment of a circle about the axis of rotation is

$$I_s = \frac{\lambda R^4}{4}\left[\theta + \sin\theta\left(\frac{2}{3}\sin^2\frac{\theta}{2} - 1\right)\right]. \qquad (88)$$

Since the moment-of-inertia of the segment of a circle about the axis of rotation is already known, it is not necessary to invoke the parallel axis theorem. Rather, the contribution to the kinetic energy from the segment of a circle can be found directly by substituting (88) into (1) with the result

$$E_s = \frac{1}{2} I_s \omega^2 = \frac{\lambda R^4}{8}\left[\theta + \sin\theta\left(\frac{2}{3}\sin^2\frac{\theta}{2} - 1\right)\right]\omega^2 . \tag{89}$$

When (10) relating ω in radians per second to Ω in rpm is substituted into (89) and the numerical value of π is applied, the result is

$$E_s = \frac{\lambda R^4}{729.5}\left[\theta + \sin\theta\left(\frac{2}{3}\sin^2\frac{\theta}{2} - 1\right)\right]\Omega^2 . \tag{90}$$

If all lengths are measured in feet and the mass of the showcase floor is expressed in "equivalent" pounds, E_s from (90) is in units of lb-ft.

Let Γ_1, Γ_2 and Γ_s equal the coefficients of Ω^2 in (86), (87) and (90), respectively. That is,

$$\Gamma_1 = \frac{\lambda A_1}{182.38}\left(\rho_1^2 + \frac{l_1^2}{18}\right), \tag{91a}$$

$$\Gamma_2 = \frac{\lambda A_2}{182.38}\left(\rho_2^2 + \frac{l_2^2}{18}\right), \tag{91b}$$

and,

$$\Gamma_s = \frac{\lambda R^4}{729.5}\left[\theta + \sin\theta\left(\frac{2}{3}\sin^2\frac{\theta}{2} - 1\right)\right]. \tag{91c}$$

The total rotational kinetic energy contributed by the two identical showcase floors is given by twice the sum of (86), (87) and (90). That is,

$$E_{floors} = 2\left(E_1 + E_2 + E_s\right) = 2\left(\Gamma_1 + \Gamma_2 + \Gamma_s\right)\Omega^2 . \tag{92}$$

The rate in rpm at which the two showcase floors would have to rotate to contribute exactly 2.5 lb-ft of kinetic energy is found from (92) by setting E_{floors} to 2.5 lb-ft and solving for Ω. The result is

$$\Omega_{2.5} = \frac{1.118}{\sqrt{\Gamma_1 + \Gamma_2 + \Gamma_s}} . \tag{93}$$

Likewise, by setting E_{floors} in (92) to 7.0 lb-ft and solving for Ω, the rate in rpm at which the two showcase floors would have to rotate to contribute exactly 7.0 lb-ft of kinetic energy is found to be

$$\Omega_{7.0} = \frac{1.871}{\sqrt{\Gamma_1 + \Gamma_2 + \Gamma_s}}\,. \tag{94}$$

Kinetic Energy Equations for Ceiling, Point & Extended Objects

Ceilings

The RDEC program enables the user to incorporate a co-rotating ceiling in the calculation of the kinetic energy of the revolving door. The following assumptions are made.

1. The ceiling is circular.

2. The distribution of the mass of the ceiling is distributed uniformly over the circular ceiling.

3. The ceiling co-rotates with the door.

4. The geometric center of the ceiling coincides with the axis of rotation of the door.

Because the center-of-mass of a circular uniformly distributed mass is located at the geometric center of the circle and, given the assumption that the center of the ceiling coincides with the axis of rotation of the door, it is not necessary to invoke the parallel axis theorem in order to derive an equation for the rotational kinetic energy contributed by a co-rotating ceiling. Rather, the derivation can begin immediately with the well known moment-of-inertia about the geometric center of a plane circular uniformly distributed mass distribution

$$I = \frac{1}{2} M R^2 \tag{95}$$

where M is the total mass of the ceiling and R is the radius of the ceiling.

The kinetic energy of the ceiling is found by substituting (95) into (1) giving

$$E_{ceiling} = \frac{1}{4} M R^2 \omega^2 \tag{96}$$

where ω is the rotation rate of the door and ceiling in radians per second.

Using (10) to relate rotation rate ω in radians per second to Ω in rpm, (9) to relate the mass M of the ceiling to its weight W in pounds, and the fact that the radius R is half the diameter D of the ceiling gives, from (96)

$$E_{ceiling} = \frac{WD^2}{16g}\left(\frac{\pi}{30}\right)^2 \Omega^2 = \frac{WD^2}{46936}\Omega^2 . \tag{97}$$

The numerical value of π and the approximate value of g at sea level of 32.17 ft/sec^2 have been substituted to form the right hand expression in (97). The units of $E_{ceiling}$ from (97) will be pound-feet (lb-ft) if the weight W of the ceiling is in pounds and the diameter D of the ceiling is in feet.

Equation (97) can be used to calculate the rotation rate Ω at which the ceiling would carry exactly 2.5 lb-ft of rotational kinetic energy by solving (97) for Ω with $E_{ceiling}$ set to 2.5 lb-ft. The result is

$$\Omega_{2.5} = \frac{343}{D\sqrt{W}} . \tag{98}$$

Likewise, equation (97) can be used to calculate the rotation rate Ω at which the ceiling would carry exactly 7.0 lb-ft of rotational kinetic energy by solving (97) for Ω with $E_{ceiling}$ set to 7.0 lb-ft. The result is

$$\Omega_{7.0} = \frac{573}{D\sqrt{W}} . \tag{99}$$

Extended Objects

In order to accommodate inclusion in the kinetic energy calculations the effects of other objects whose kinetic energy equations have not been included within the RDEC program explicitly, RDEC allows the user to characterize such additional objects by their total mass, expressed by weight at sea level, the location within the door of their centers-of-mass and, if significant, their moments-of-inertia about their own centers-of-mass. In this way, the effects on the overall kinetic energy of the door due to virtually any additional co-rotating object or objects can be accounted for properly. When the moment-of-inertia of an additional object about its own center-of-mass is significant, it is referred to as an "extended object" in the nomenclature of RDEC.

Calculation of the rotational kinetic energy starts in this case with the parallel axis theorem (2)

$$I = Mr^2 + I_{cm} \tag{2}$$

where I is the total moment-of-inertia of the object about the axis of rotation, M is the total mass of the extended object, r is the radial distance of the center-of-mass of the object from the axis of rotation, and I_{cm} is the moment-of-inertia of the object about its own center-of-mass, also referred to as the "self" moment-of-inertia of the object.

The rotational kinetic energy of such an object is found by substituting its moment-of-inertia from (2) into the general expression for rotational kinetic energy (1) to get

$$E_{ext'd} = \frac{1}{2}\left(Mr^2 + I_{cm}\right)\omega^2 . \tag{100}$$

Using (10) to relate rotation rate ω in radians per second to Ω in rpm and (9) to relate both the mass M of the object to its weight W in pounds as well as its self moment-of-inertia I_{cm} in slug-ft^2 to pound-ft^2 gives, from (100) [2]

$$E_{ext'd} = \frac{1}{2g}\left(Wr^2 + I_{cm}\right)\left(\frac{\pi}{30}\right)^2 \Omega^2 = \frac{Wr^2 + I_{cm}}{5867}\Omega^2 . \tag{101}$$

The numerical value of π and the approximate value of g at sea level of 32.17 ft/sec^2 have been substituted to form the right hand expression in (101). The units of $E_{ext'd}$ from (101) will be pound-feet (lb-ft) if the weight W of the extended object is in pounds and the distance r is in feet. Note that I_{cm} is assumed to be in units of slug-ft^2 in (100) but is in units of pound-ft^2 in (101), although the identical symbol has been used in both expressions.

Equation (101) can be used to calculate the rotation rate Ω at which the extended object would carry exactly 2.5 lb-ft of rotational kinetic energy by solving (101) for Ω with $E_{ext'd}$ set to 2.5 lb-ft. The result is

$$\Omega_{2.5} = \frac{121}{\sqrt{I_{cm} + Wr^2}} . \tag{102}$$

Likewise, equation (101) can be used to calculate the rotation rate Ω at which the extended object would carry exactly 7.0 lb-ft of rotational kinetic energy by solving (101) for Ω with $E_{ext'd}$ set to 7.0 lb-ft. The result is

---

[2] Note that both the mass M and the moment-of-inertia I_{cm} are divided by g in order to be expressed in terms of weight in pounds as a proxy for the mass in slugs.

$$\Omega_{7.0} = \frac{203}{\sqrt{I_{cm} + Wr^2}} \,. \tag{103}$$

Point Objects

When the moment-of-inertia of an object to be included in the rotational kinetic energy calculations of the door is negligible, the RDEC program allows the user to characterize such additional objects by their total mass, expressed by weight at sea level, and the location within the door of their centers-of-mass. In this way, the effects on the overall kinetic energy of the door due to any such additional co-rotating objects can be accounted for properly. When the moment-of-inertia of an additional object about its own center-of-mass is not significant and can be ignored, it is referred to as a "point object" in the nomenclature of RDEC.

Since the only difference between "extended objects" and "point objects" is that the moments-of-inertia of the latter about their own centers-of-mass are not significant and can safely be ignored, the kinetic energy equations pertaining to point objects can be obtained immediately from those already derived for extended objects merely by omitting I_{cm} from the relevant expressions. Thus, from (101)

$$E_{point} = \frac{Wr^2}{5867}\Omega^2 \tag{104}$$

where W is the weight in pounds of the point object, r is the radial distance of its center-of-mass from the axis of rotation, and Ω is the rotation rate of the door in rpm. With these units for W and r, E_{point} is in units of pound-feet (lb-ft).

The rate at which the point object would have to rotate in order to carry exactly 2.5 lb-ft of kinetic energy is found from (102) by omitting I_{cm} and is

$$\Omega_{2.5} = \frac{121}{r\sqrt{W}} \,. \tag{105}$$

And likewise, the rate at which the point object would have to rotate in order to carry exactly 7.0 lb-ft of kinetic energy is found from (103) by omitting I_{cm} and is

$$\Omega_{7.0} = \frac{203}{r\sqrt{W}} \,. \tag{106}$$

Appendix 1

Center-of-mass of a Right Triangle

The vector location of the center-of-mass of any mass distribution is defined as

$$\overline{R} = \frac{1}{M}\int \overline{r}\, dm \qquad (1)$$

where $\overline{R}$ is the vector location of the center-of-mass and M is the total mass of the mass distribution.

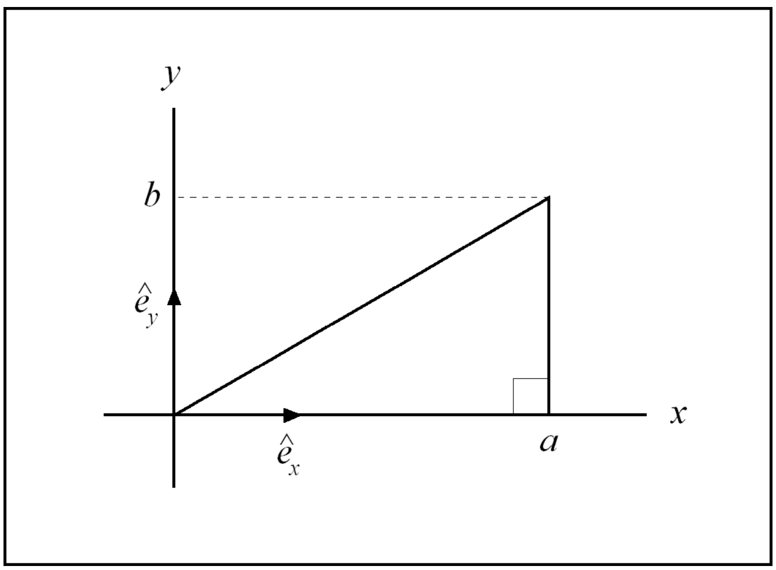

Figure 1. Right triangle with perpendicular sides of lengths *a* and *b*.

Assume that the area mass density of the triangle is uniform with value λ. The total mass of the right triangle is the product of λ and its area. That is

$$M = \frac{1}{2}\lambda\, a\, b\,. \qquad (2)$$

The differential mass element dm is

$$dm = \lambda\, dy\, dx \qquad (3)$$

and the vector location of the differential mass element in the integrand of (1) is

$$\bar{r} = \hat{e}_x x + \hat{e}_y y \tag{4}$$

where $\hat{e}_x$ and $\hat{e}_y$ are unit vectors in the *x* and *y* directions, respectively. The equation describing the hypotenuse of the right triangle in Figure 1 is

$$y = \frac{b}{a} x\ . \tag{5}$$

From (1) and the above definitions, the center-of-mass of the right triangle illustrated in Figure 1 is

$$\bar{R} = \frac{\lambda}{M} \int_0^a \int_0^{\frac{b}{a}x} \left(\hat{e}_x x + \hat{e}_y y \right) dy\, dx =$$

$$= \hat{e}_x \frac{\lambda}{M} \int_0^a x \int_0^{\frac{b}{a}x} dy\, dx + \hat{e}_y \frac{\lambda}{M} \int_0^a \int_0^{\frac{b}{a}x} y\, dy\, dx\ . \tag{6}$$

Evaluating the inner integrals on *y* in (6),

$$\bar{R} = \hat{e}_x \frac{\lambda}{M} \frac{b}{a} \int_0^a x^2 dx + \hat{e}_y \frac{\lambda}{M} \frac{b^2}{2a^2} \int_0^a x^2 dx\ . \tag{7}$$

And evaluating the remaining integrals on *x* in (7),

$$\bar{R} = \hat{e}_x \frac{\lambda}{M} \frac{b}{a} \frac{a^3}{3} + \hat{e}_y \frac{\lambda}{M} \frac{b^2}{2a^2} \frac{a^3}{3}\ . \tag{8}$$

When (2) for *M* is inserted into (8) and the fractions are cleared, the final result is

$$\bar{R} = \hat{e}_x \frac{2}{3} a + \hat{e}_y \frac{1}{3} b\ . \tag{9}$$

In words, the center-of-mass of the right triangle illustrated in Figure 1 is located at a distance (2/3) a from the origin along the side of length a, or, equivalently, (1/3) a from the right angle of the triangle. Likewise, the location of the center-of-mass is at a distance of (1/3) b from the origin along the side of length b which, in this instance, is also (1/3) b from the right angle of the triangle. That is, the center-of-mass of a right triangle of perpendicular sides of length a and b is located at a distance of (1/3) a along the side of length a from the right angle, and at a distance of (1/3) b along the side of length b from the right angle, as illustrated in the Figure below.

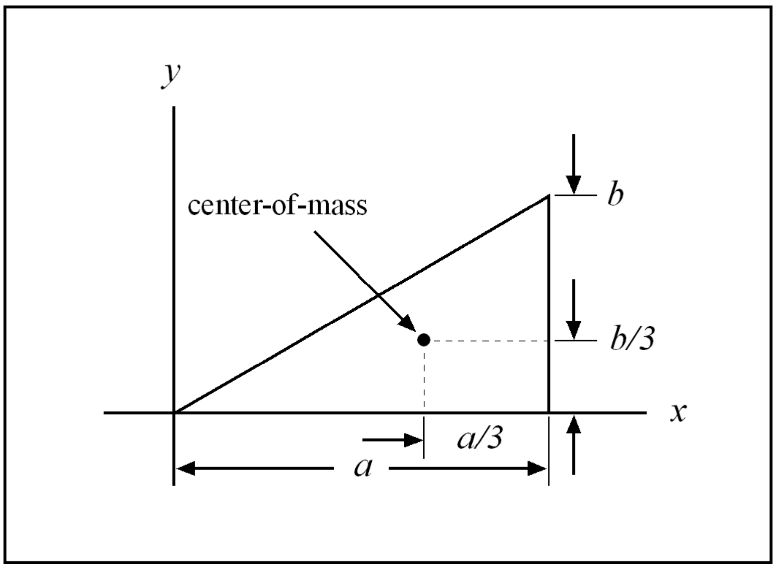

Figure 2. Center-of-mass of a right triangle relative to the location of the right angle.

Appendix 2

Moment-of-inertia of a Right Triangle

The so-called polar moment-of-inertia of a mass distribution is defined about the origin of coordinates by

$$I = \int r^2 dm \tag{1}$$

where I is the moment-of-inertia, dm is the differential mass element, and r^2 is the square of the distance of the differential mass element from the origin.

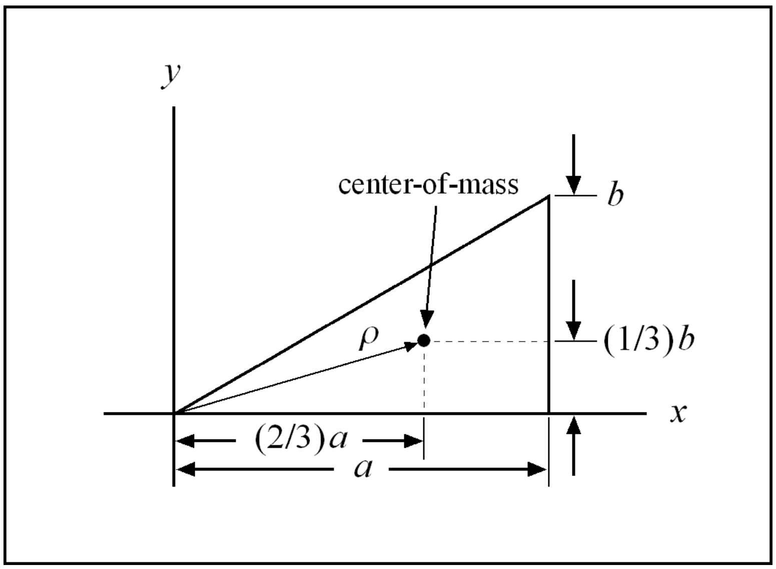

Figure 1. Right triangle with perpendicular sides of lengths a and b, with center-of-mass location as derived in Appendix 1.

Assuming that the triangle has uniform area mass distribution λ, the differential mass element dm is

$$dm = \lambda \, dy \, dx \, . \tag{2}$$

The square of the distance of the differential mass element from the origin with respect to which the moment-of-inertia is to be calculated is

$$r^2 = x^2 + y^2 \,. \tag{3}$$

The equation describing the hypotenuse of the right triangle illustrated in Figure 1 is

$$y = \frac{b}{a} x \,. \tag{4}$$

From (1) and the above definitions, the moment-of-inertia of the illustrated right triangle about the origin of coordinates in Figure 1 is

$$I_0 = \lambda \int_0^a \int_0^{\frac{b}{a}x} \left(x^2 + y^2 \right) dy\, dx =$$

$$= \lambda \int_0^a x^2 \int_0^{\frac{b}{a}x} dy\, dx + \lambda \int_0^a \int_0^{\frac{b}{a}x} y^2 dy\, dx \,. \tag{5}$$

Evaluating the inner integrals on y in (5),

$$I_0 = \lambda \frac{b}{a} \int_0^a x^3 dx + \lambda \frac{b^3}{3a^3} \int_0^a x^3 dx \,. \tag{6}$$

And, evaluating the remaining integrals on x in (6),

$$I_0 = \lambda \frac{b}{a} \frac{a^4}{4} + \lambda \frac{b^3}{3a^3} \frac{a^4}{4} = \frac{\lambda\, a\, b}{12} \left(3a^2 + b^2 \right) . \tag{7}$$

I_0 is the moment-of-inertia of the right triangle illustrated in Figure 1 about the origin, which is coincident with the vertex at the left of the triangle. The moment-of-inertia about the center-of-mass I_{cm} of the triangle can be found by applying the parallel axis theorem, equation (2) on page 5 in the *Introduction* to this document. Since the area mass density λ of the triangle is assumed to be constant over the triangle, the total mass M of the triangle is the product of λ and the area of the triangle. That is,

$$M = \frac{1}{2} \lambda\, a\, b \,. \tag{8}$$

Substituting (8) for M into equation (2) on page 5 of the *Introduction* and with ρ in Figure 1 playing the role of r in (2), and solving for I_{cm},

$$I_{cm} = I_0 - \frac{1}{2}\lambda\, a\, b \rho^2 . \quad (9)$$

From Figure 1 and the Pythagorean theorem, the squared length of the displacement ρ of the center-of-mass of the triangle from the origin is

$$\rho^2 = \left(\frac{2}{3}a\right)^2 + \left(\frac{1}{3}b\right)^2 = \frac{4a^2 + b^2}{9} . \quad (10)$$

Substituting (7) and (10) into (9) gives

$$I_{cm} = \frac{\lambda\, a\, b}{12}\left(3a^2 + b^2\right) - \frac{\lambda\, a\, b}{18}\left(4a^2 + b^2\right) = \frac{\lambda\, a\, b}{36}\left(a^2 + b^2\right). \quad (11)$$

And, finally, substituting M from (8) into (11) gives for the polar moment-of-inertia about the center-of-mass of a right triangle with perpendicular sides a and b and uniform area mass density

$$I_{cm} = \frac{M}{18}\left(a^2 + b^2\right) \quad (12)$$

where M is the total mass of the right triangle. Note that $a^2 + b^2$ in (12) is equal to the square of the hypotenuse of the right triangle.

Appendix 3

Moment-of-inertia of a Segment of a Circle

The moment-of-inertia of a mass distribution about the origin of coordinates is

$$I = \int r^2 dm \tag{1}$$

where *I* is the moment-of-inertia, *dm* is the differential mass element, and r^2 is the square of the distance of the differential mass element from the origin.

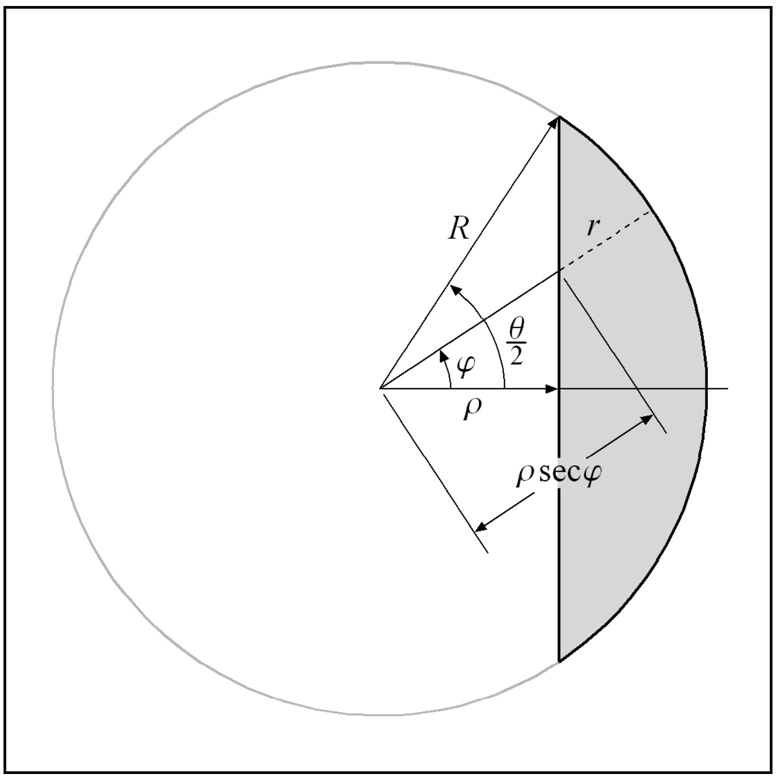

Figure 1. Variables and dimensions used to integrate over the segment of a circle.

Assuming that the segment of a circle has uniform area mass density λ, in polar coordinates with origin at the center of the circle, the differential mass element *dm* is

$$dm = \lambda\, r\, dr\, d\varphi\,. \tag{2}$$

The distance ρ in Figure 1 is

$$\rho = R\cos\frac{\theta}{2} \tag{3}$$

where θ is the total angle about the center of the circle subtended by the chord that defines the segment of the circle.

From (1) and the above definitions, the moment-of-inertia of the segment of a circle about the center of the circle in Figure 1 is

$$I = 2\lambda\int_0^{\frac{\theta}{2}}\int_{\rho\sec\varphi}^{R} r^3 dr\, d\varphi . \tag{4}$$

Due to symmetry, the integration is carried out only over the upper half of the segment of a circle with the factor of 2 appearing on the right side of (4) accounting for the entire segment. That is, the sum of the integrations over both the upper and lower halves of the segment.

The inner integral on r in (4) evaluates to

$$\frac{1}{4}\left(R^4 - \rho^4 \sec^4\varphi\right)$$

so that (4) breaks into the sum of two integrals

$$I = \frac{\lambda R^4}{2}\int_0^{\frac{\theta}{2}} d\varphi - \frac{\lambda\rho^4}{2}\int_0^{\frac{\theta}{2}}\sec^4\varphi\, d\varphi . \tag{5}$$

The second integral in (5) is evaluated as follows[3]

$$\int_0^{\frac{\theta}{2}}\sec^4\varphi\, d\varphi \quad = \quad \int_0^{\frac{\theta}{2}}\frac{d\varphi}{\cos^4\varphi} \quad = \quad \left[\frac{\sin\varphi}{3\cos^3\varphi} + \frac{2}{3}\frac{\sin\varphi}{\cos\varphi}\right]_0^{\frac{\theta}{2}} \quad =$$

[3] See *Table of Integrals, Series and Products*, I. S. Gradshteyn and I. M. Ryzhik, Integral 2.526(12), Academic Press (1980).

$$= \left[\frac{\sin\varphi\cos\varphi + 2\sin\varphi\cos^3\varphi}{3\cos^4\varphi} \right]_0^{\frac{\theta}{2}} = \frac{\sin\frac{\theta}{2}\cos\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos^3\frac{\theta}{2}}{3\cos^4\frac{\theta}{2}}. \tag{6}$$

Use the trigonometric identities

$$\sin 2x = 2\sin x\cos x \quad \text{and} \quad \cos^2 x = 1 - \sin^2 x$$

in (6) to get

$$\int_0^{\frac{\theta}{2}} \sec^4\varphi d\varphi = \frac{\frac{1}{2}\sin\theta + \sin\theta\cos^2\frac{\theta}{2}}{3\cos^4\frac{\theta}{2}} =$$

$$= \frac{\frac{3}{2}\sin\theta - \sin\theta\sin^2\frac{\theta}{2}}{3\cos^4\frac{\theta}{2}}. \tag{7}$$

With (3) for ρ and result (7), the second integral in (5) becomes

$$\frac{\lambda\rho^4}{2}\int_0^{\frac{\theta}{2}} \sec^4\varphi d\varphi = \frac{\lambda R^4}{4}\left(\sin\theta - \frac{2}{3}\sin\theta\sin^2\frac{\theta}{2}\right). \tag{8}$$

The first integral in (5) evaluates to

$$\frac{\lambda}{2}R^4\int_0^{\frac{\theta}{2}} d\varphi = \frac{\lambda R^4}{4}\theta, \tag{9}$$

and when (8) and (9) are used in (5) the result is finally

$$I = \frac{\lambda R^4}{4}\left[\theta + \sin\theta\left(\frac{2}{3}\sin^2\frac{\theta}{2} - 1\right)\right]. \tag{10}$$

Appendix 4

Moment-of-inertia of Core Floor

Herein are derived the equations for the moment-of-inertia of core floors or independent core ceilings. Although equations for the kinetic energy of a core floor or ceiling do not appear in the RDEC program, the required moment-of-inertia equations do appear in the RDEC User's Manual in connection with an example of the use of the RDEC Extended object option. Therefore, for completeness, their derivations are provided here.

By "independent core ceiling" it is meant that the core ceiling does not consist of the central part of a circular co-rotating ceiling already being accounted for using the Ceiling option available in the RDEC Configuration window but, rather, that the ceiling, if it exists, is actually an independent structure whose effect is to be included in the kinetic energy calculations.

In the following, the term "core floor" should be understood to mean any structure, including a ceiling, exhibiting the following characteristics.

1. The surface of the structure is oriented horizontally and is bounded on all sides by the flat core panels. Thus, the structure has the same size and shape as the core.

2. The core shape is that of a n-sided regular polygon.

3. The distribution of mass over the surface of the structure is uniform.

4. The distribution of the mass of the structure in the vertical direction is immaterial.

5. The geometric center of the n-sided regular polygon coincides with the axis of rotation of the door.

Figure 1 below illustrates a core floor corresponding the a regular pentagon. A 5-sided core floor has been chosen for pedagogical purposes to highlight certain aspects of the nature of the derivation and is not meant necessarily to represent a shape that might be encountered in actual practice. It is clear from the Figure that

$$a = r\cos\frac{\theta}{2} \qquad (1)$$

and

$$b = r\sin\frac{\theta}{2} \tag{2}$$

Note that the sector of the polygon highlighted in Figure 1 is comprised of two identical right triangles with perpendicular sides *a* and *b*. Equation (7) in Appendix 2 contains the following expression for the moment-of-inertia about the acute vertex located at the origin of coordinates of a right triangle with perpendicular sides *a* and *b*

$$I_0 = \frac{\lambda ab}{12}\left(3a^2 + b^2\right) \tag{3}$$

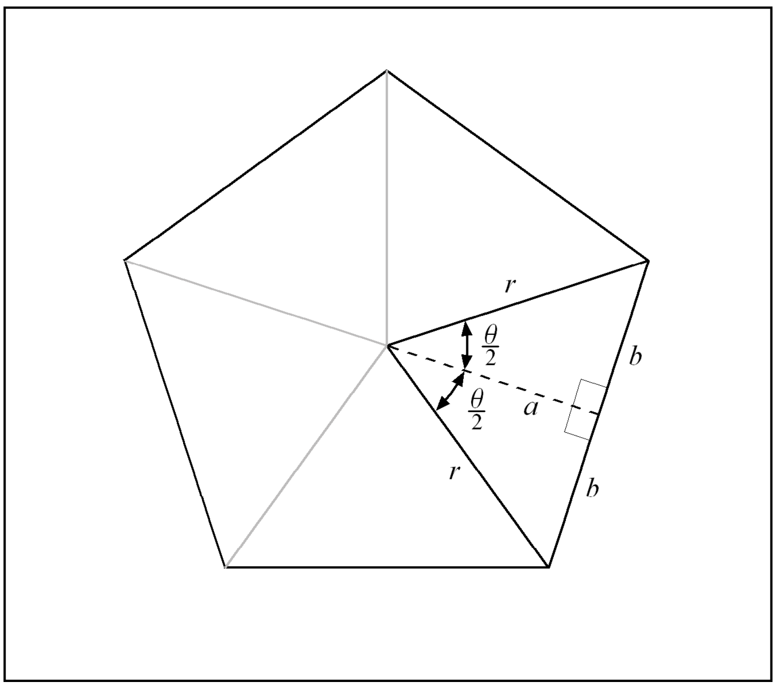

Figure 1. Illustration of a regular polygonal core floor with $n = 5$.

where λ is the area mass density of the triangle, which is assumed to be uniform. Assuming the origin of coordinates in Figure 1 to be at the geometric center of the polygon, equation (3) correctly describes the moment-of-inertia about the center of the polygon of each of the two triangles comprising the polygon sector. Since identical pairs of triangles exhibiting the same moment-of-inertia as given by (3) can be constructed within each of the *n* sectors of the *n*-sided polygon, the total moment-of-inertia about the geometric center of the entire polygon is

$$I_n = 2nI_0 = \frac{n\lambda ab}{6}\left(3a^2 + b^2\right). \tag{4}$$

Accounting for both triangles within the sector, the area of each sector of the polygon is

$$A = 2\left(\frac{1}{2}ab\right) = ab. \tag{5}$$

Given that there are *n* sectors in an *n*-sided polygon and given the uniform area mass density λ, the total mass of the polygon is

$$M = n\lambda A = n\lambda ab. \tag{6}$$

With (6), equation (4) for the moment-of-inertia of the polygon can be expressed in terms of the mass of the polygon as

$$I_n = \frac{M}{6}\left(3a^2 + b^2\right). \tag{7}$$

With *a* and *b* from (1) and (2) in (7), the moment-of-inertia I_n is

$$I_n = \frac{Mr^2}{6}\left(3\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right) = \frac{Mr^2}{6}\left(2\cos^2\frac{\theta}{2} + 1\right) \tag{8}$$

where the trigonometric identity $\sin^2 x = 1 - \cos^2 x$ has been used in (8). And, with the further trigonometric identity

$$2\cos^2 x = 1 + \cos 2x$$

used in (8),

$$I_n = \frac{Mr^2}{6}\left(2 + \cos\theta\right) = \frac{Mr^2}{6}\left(2 + \cos\frac{2\pi}{n}\right) \tag{9}$$

where the fact that the central angle θ subtended by a sector of an *n*-sided polygon is

$$\theta = \frac{2\pi}{n} \tag{10}$$

has also been used in (9).

For the specific cases of a 3-sided (equilateral triangle) and a 4-sided (square) regular polygonal core floor, the respective moments-of-inertia are found from (9) by substituting $n = 3$ and $n = 4$. The results are

$$I_3 = \frac{1}{4} M r^2 \tag{11}$$

and

$$I_4 = \frac{1}{3} M r^2 \tag{12}$$

where I_3 and I_4 are the moments-of-inertia about the geometric centers of a 3-sided and a 4-sided regular polygonal core floor, respectively, M is the mass of the floor, and r is the distance from the geometric center of the polygon to any vertex of the polygon.

Because the geometric center of the polygonal floor is assumed to coincide with the axis of rotation of the door, (11) and (12) also express the moments-of-inertia of the respective core shapes about the axis of rotation of the door.

For the purpose of entering, in English units, the moments-of-inertia given by (11) and (12) as one of the parameters required by the Extended objects option of the RDEC program, the weight W of the core floor, rather than its true mass M, should be substituted for M. That is, when using English units with the RDEC Extended objects option, the following should be used

$$I_3 = \frac{1}{4} W r^2 \tag{13}$$

and

$$I_4 = \frac{1}{3} W r^2 \tag{14}$$

where I_3 and I_4 are the moments-of-inertia about the geometric centers of a 3-sided and a 4-sided regular polygonal core floor in units of lb-ft^2, respectively, W is the weight of the floor in pounds, and r is the distance from the geometric center of the polygon to any vertex of the polygon in feet.

Note that the substitution of the weight W for the true mass M is required only when using English units. Otherwise, when using metric units, equations (11) and (12) should be used with M specified in kilograms and r in meters so that I_3 and I_4 will be in units of kg-m^2.

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